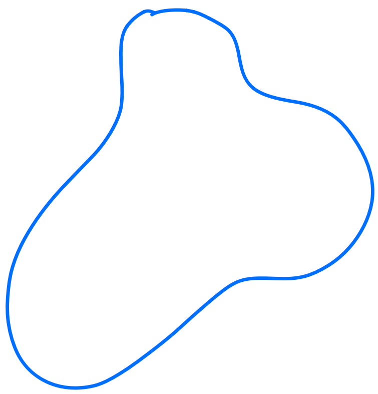
Holomorphic Dynamics – Lecture 15

Local Connectivity

Carathéodory theory (1913)

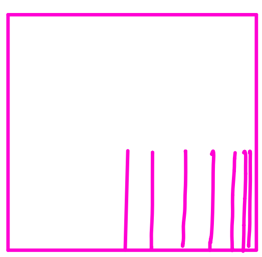


$\mathcal{U} \subsetneq \mathbb{C}$
simply connected and open

Then there exists $\varphi : \mathcal{U} \longrightarrow \mathbb{D}$ which is biholomorphic

$\boxed{Q}$ Does φ extend to $\partial \mathcal{U}$?
Does φ^{-1} extend to $\partial \mathbb{D}$?

$\mathcal{U} =$



$\partial \mathcal{U}$ is <u>not</u> homeo to $\partial \mathbb{D}$

Case we're interested in:

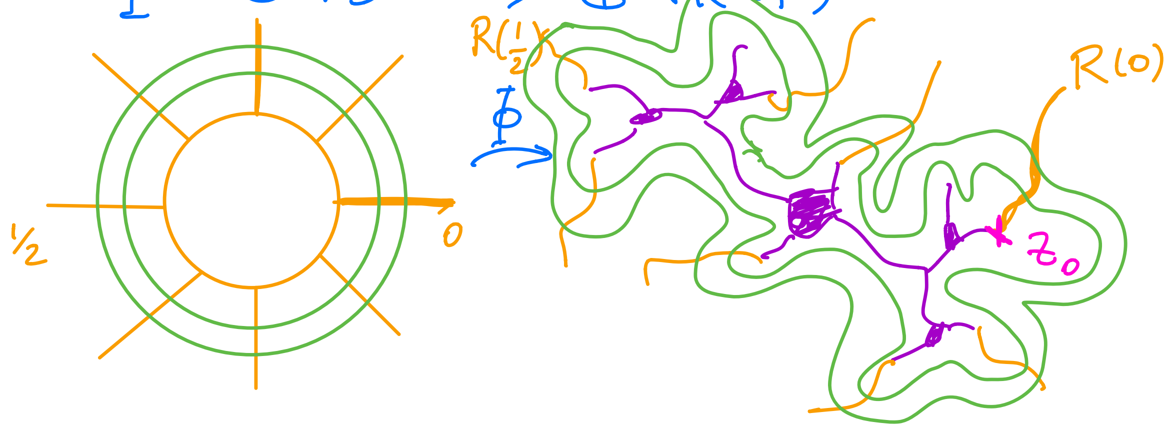
$f : \mathbb{C} \to \mathbb{C}$ polynomial of degree d

Def.: $K(f) := \{ z : (f^n(z))_{n\geq 0} \text{ is bounded} \}$

If $K(f)$ is connected, $\hat{\mathbb{C}} \setminus K(f) \simeq \mathbb{D}$

Böttcher map:

$$\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \longrightarrow \hat{\mathbb{C}} \setminus K(f)$$



Def.: Given $\theta \in \mathbb{R}/\mathbb{Z}$, the <u>external ray</u> at angle θ is

$$R(\vartheta) := \{ \Phi(r e^{2\pi i \theta}), \ r > 1 \}$$

<u>In fact</u>: Φ semiconjugates $z \mapsto z^d$ to f

$$\begin{tikzcd} \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "\Phi"] \arrow[loop below, "z \mapsto z^d"] & \hat{\mathbb{C}} \setminus K(f) \arrow[loop below, "f"] \end{tikzcd} \qquad \Phi(z^d) = f(\Phi(z))$$

Def.: We say that $R(\theta)$ <u>lands</u> if

$$\lim_{r \to 1^+} \Phi(re^{2\pi i \theta}) \quad \text{exists.}$$

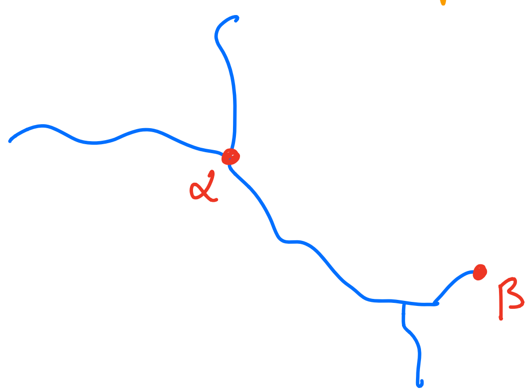
$\boxed{Q}$ For what f and what θ does $R(\theta)$ land?

E.g.: if $R(0)$ lands at z_0

then $R(d \cdot 0)$ lands at $f(z_0)$

hence $f(z_0) = z_0$

Def.: The landing point of $R(0)$ is called the β fixed point.



Suppose that $R(\theta)$ lands for <u>all</u> θ.
Then we have defined

$$\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow J(f)$$

$$\gamma(\theta) := \lim_{r \to 1^+} \Phi(re^{2\pi i \theta})$$

Def.: γ is called <u>Carathéodory</u> loop and satisfies

$$\gamma(d\theta) = f(\gamma(\theta))$$

Cor.: if θ has period p under $\theta \mapsto d\theta \mod 1$,

then $\gamma(\theta)$ has period dividing p.

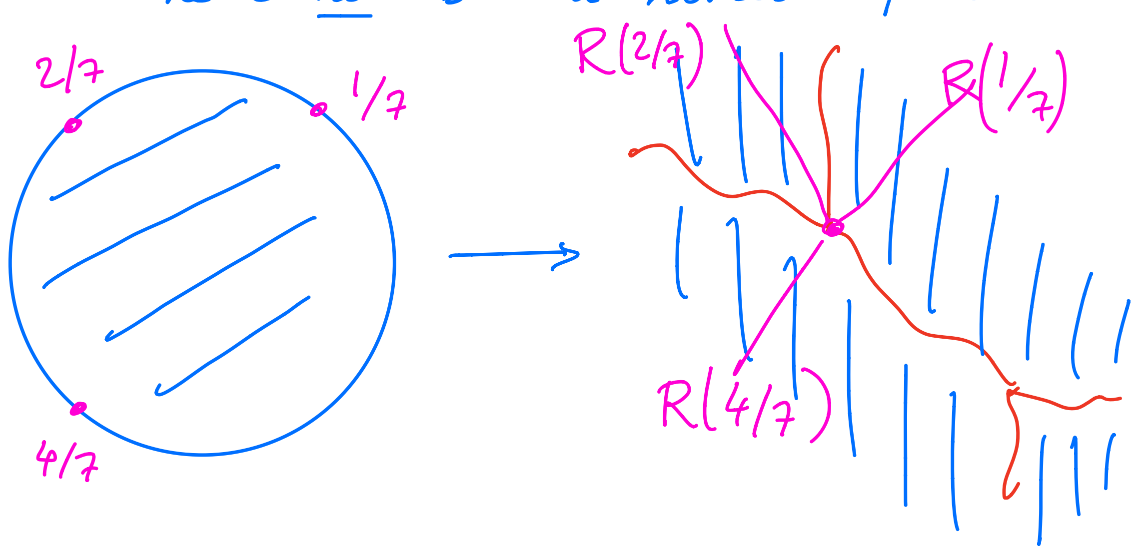
Pf.:

$$d^p\theta = \theta \pmod 1$$

$$\gamma(d^p\theta) = \gamma(\theta)$$

$$\| \quad f^p(\gamma(\theta))$$

Note: $\gamma: \mathbb{R}/\mathbb{Z} \simeq S^1 \longrightarrow J(f)$ need <u>not</u> be a homeomorphism.



Theorem There exist polynomials for which not all rays land (Milnor)

Theorem All rational rays land.

Theorem If f is hyperbolic, then all rays land.

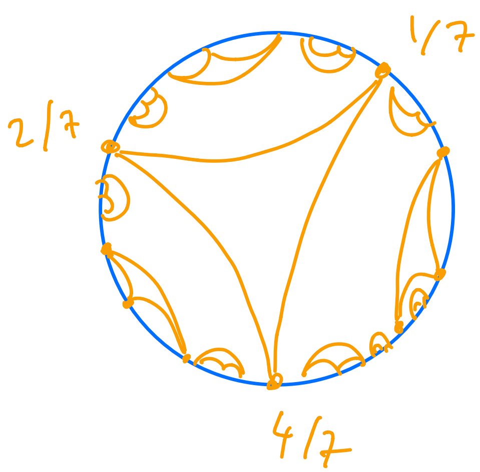
Prop.: If the Carathéodory loop γ is continuous, then we can define an equivalence relation $\sim$ on $\mathbb{R}/\mathbb{Z}$ as

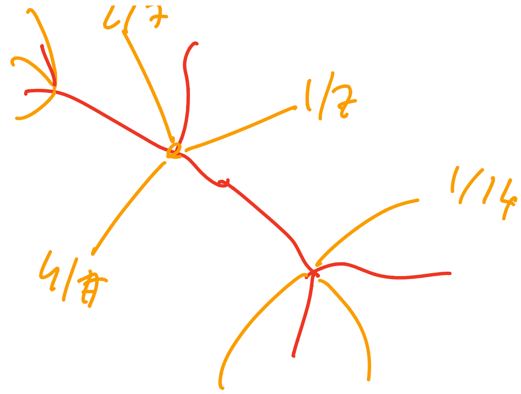
$\theta_1 \sim \theta_2$ if $R(\theta_1)$ lands at same point as $R(\theta_2)$

and γ induces a homeomorphism

$$\mathbb{R}/\mathbb{Z}\big/\sim \;\xrightarrow{\;\sim\;}\; J(f)$$

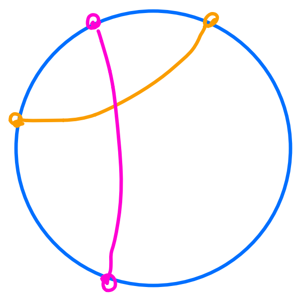
Note: By connecting the angles θ with the same landing point on $\partial\mathbb{D}$ by hyperbolic geodesics in $\mathbb{D}$, we obtain a <u>lamination</u>

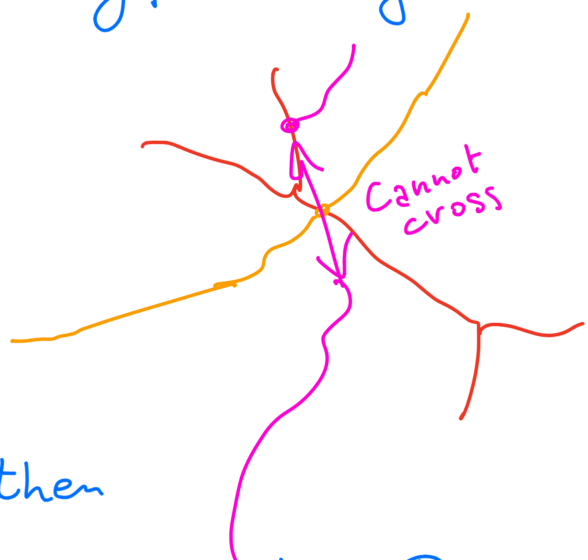




Def: A <u>lamination</u> of $\mathbb{D}$ is a closed union of disjoint hyperbolic geodesics in $\mathbb{D}$.

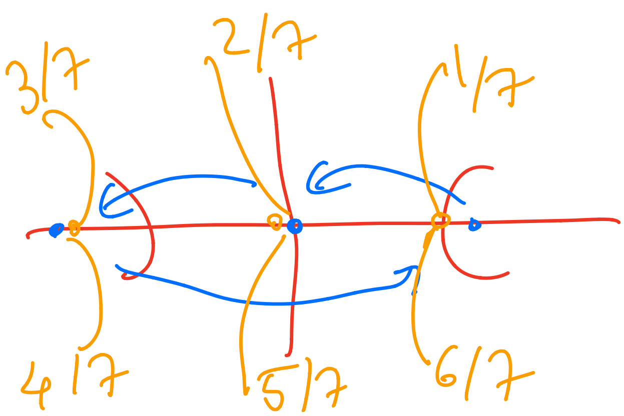
<u>disjointness</u>

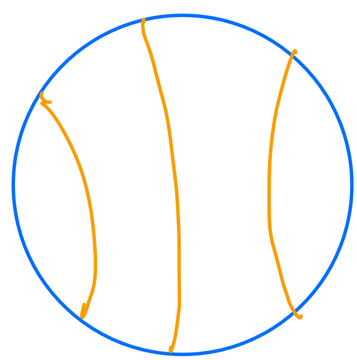




If $\theta_1 \sim \theta_2$ then

$\overline{R(\theta_1)} \cup \overline{R(\theta_2)}$ disconnects $\mathbb{C}$.





Equipotential lines

Def.: For $z \in \mathbb{C}$,

$$G(z) := \lim_{n\to\infty} \frac{\log^+ |f^n(z)|}{d^n}$$

potential function

$$G(z) > 0 \iff z \notin K(f)$$

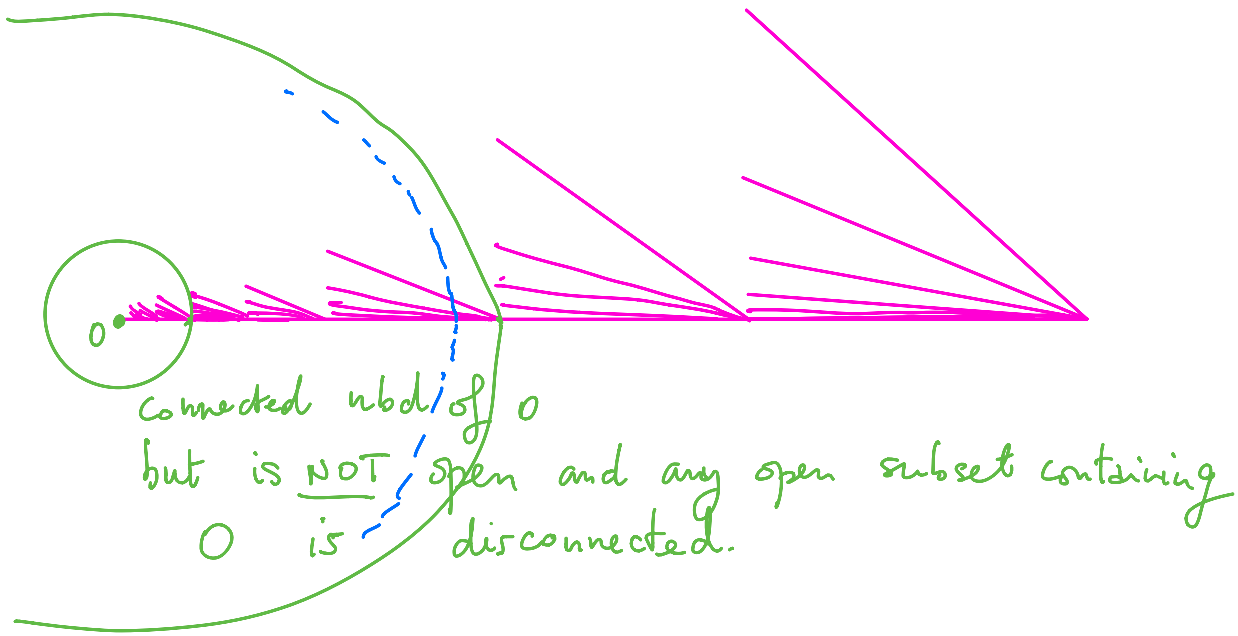
$$\log^+(x) := \max\{\log(x), 0\}$$

The lines $\{G(z) = c\}$ for $c > 0$ are called equipotential lines and are transverse to external rays.

Def.: A top. space X is LOCALLY CONNECTED at x if there exist arbitrarily small connected neighbourhoods of x.

Note we do not assume the nbds to be open, but just to contain an open containing x.

Rmk There exist spaces which are locally connected but are not openly locally connected.



Lemma X is locally connected at every $x \in X$ if and only if every open subset of X is a union of connected open subsets.

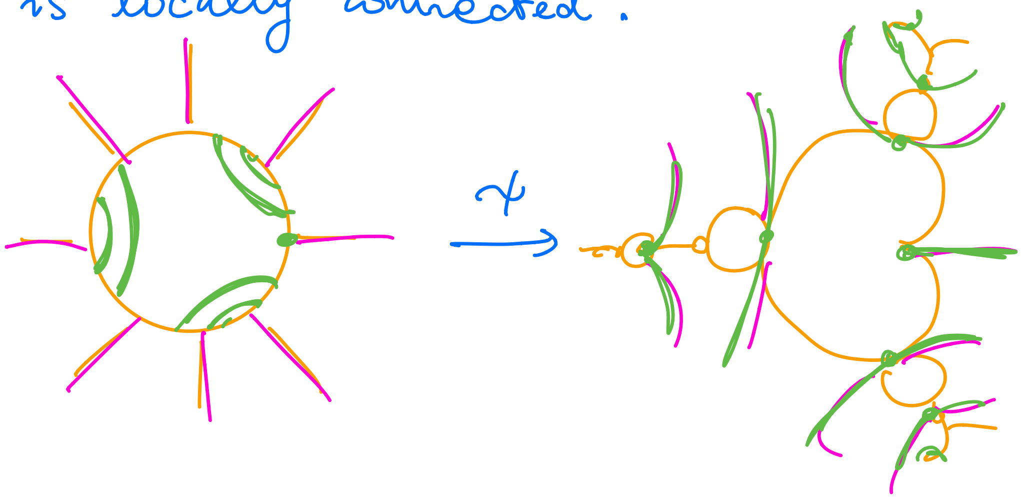
Lemma If a compact metric space X is locally connected, then it is locally path connected. Hence every connected component of X is path connected.

Lemma Any continuous image of a compact locally connected space is compact and locally connected.

Conjecture (MLC)

The Mandelbrot set is locally connected.

Thm (Carathéodory) The inverse Riemann map $\psi : \mathbb{D} \to \mathcal{U}$ extends continuously to a map from the closed disk onto $\overline{\mathcal{U}}$ if and only if the boundary $\partial \mathcal{U}$ is locally connected.

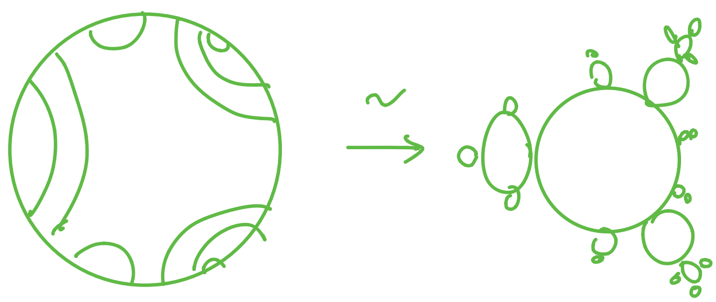


If ∂M is locally connected, then

$$\psi : \hat{\mathbb{C}} \setminus \mathbb{D} \to (\hat{\mathbb{C}} \setminus M) \cup \partial M$$

and it induces an identification

$$\hat{\psi} : S^1/\sim \;\xrightarrow{\sim}\; \partial M ,$$



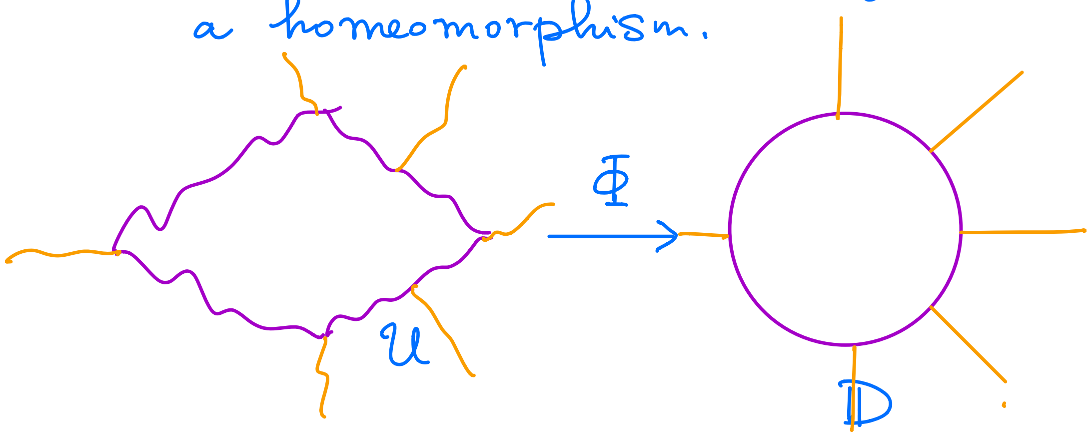
PINCHED DISK MODEL

"CACTUS MODEL"

If MLC holds, $\hat{\varphi}$ is a homeomorphism between a quotient of the circle and ∂M.

Theorem If the boundary of $\mathcal{U}$ is a Jordan curve, the Riemann map extends to a homeomorphism $\overline{\mathcal{U}} \longrightarrow \overline{\mathbb{D}}$.

E.g.: if c is in the main cardioid, then $J(f_c)$ is a Jordan curve, hence the lamination is empty and the Carathéodory loop is a homeomorphism.



Thm (Beurling) The set of θ s.t. the ray at angle θ does NOT land has Hausdorff dimension 0 (in particular it has measure 0).